

Section 6.3: The natural Exponential Function

Recall that $\ln(x)$ is a strictly increasing function for $x > 0$.

And so, $\ln(x)$ has an inverse function, denoted by $\exp(x)$.

(strictly increasing \Rightarrow one-to-one). \exp is defined by

$$\exp(x) = y \iff \ln(y) = x$$

The cancellation equations are

$$\exp(\ln(x)) = x, \text{ and } \ln(\exp(x)) = x.$$

Examples: $\ln(1) = 0 \Rightarrow \exp(0) = 1$

$$\ln(e) = 1 \Rightarrow \exp(1) = e$$

Moreover, observe that

$$\ln(e^r) = r \cdot \ln(e) = r \cdot 1 = r; \text{ Thus,}$$

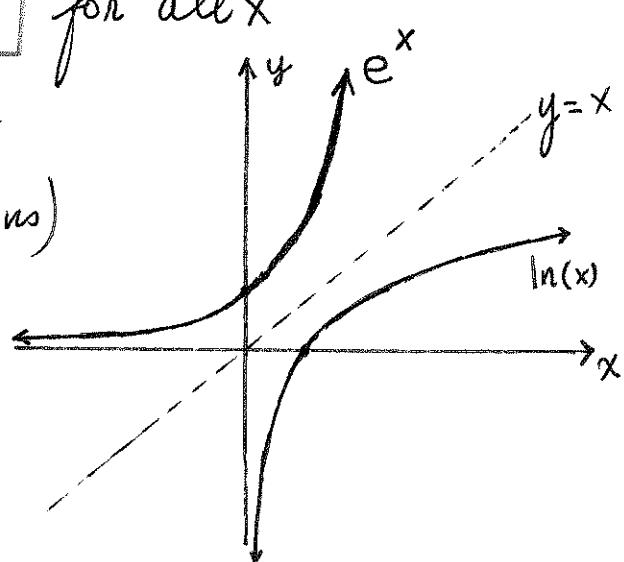
$$\ln(e^r) = r \Rightarrow \exp(r) = e^r.$$

This tells us that $\boxed{\exp(x) = e^x}$ for all x

Graph of e^x : is the reflection of the graph of $\ln(x)$ about $y=x$ (inverse functions)

Range of $\ln(x) = \mathbb{R}$ (all real numbers)

\Rightarrow domain of $e^x = \mathbb{R} = (-\infty, \infty)$



Examples: Solve for x :

$$\textcircled{1} \quad \ln(x) = 2 \Rightarrow e^{\ln(x)} = e^2 \Rightarrow x = e^2$$

$$\textcircled{2} \quad e^{x^2-1} = 1 \Rightarrow \ln(e^{x^2-1}) = \ln(1) \Rightarrow x^2-1 = 0 \Rightarrow x = \pm 1$$

$$\textcircled{3} \quad e^{6-2x} = 9 \Rightarrow \ln(e^{6-2x}) = \ln(9) \Rightarrow 6-2x = \ln(9)$$
$$\Rightarrow x = \frac{1}{2}(6 - \ln 9)$$

Properties of e^x

- e^x is an increasing function for all $x \in (-\infty, \infty)$.
- Range of e^x is $(0, \infty)$ (since domain of $\ln(x)$ is $(0, \infty)$).
Thus $e^x > 0$ for all x .
- $\lim_{x \rightarrow -\infty} e^x = 0$ (x -axis is an Asymptote at $-\infty$), and

$$\lim_{x \rightarrow \infty} e^x = \infty$$

Example: $\lim_{x \rightarrow \infty} \frac{e^{5x}}{e^{5x} + 2} = \lim_{x \rightarrow \infty} \frac{e^{5x}}{e^{5x}(1 + 2/e^{5x})} = \lim_{x \rightarrow \infty} \frac{1}{1 + \cancel{2/e^{5x}}} = 1.$

Laws of Exponents: let $x, y \in \mathbb{R}$, and r be a rational number

$$\textcircled{1} \quad e^{x+y} = e^x e^y$$

$$\textcircled{2} \quad e^{x-y} = \frac{e^x}{e^y}$$

$$\textcircled{3} \quad (e^x)^r = e^{x \cdot r}$$

Proof of ①: $\ln(e^{x+y}) = x+y$, and

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x+y. \text{ Thus}$$

$\ln(e^{x+y}) = \ln(e^x e^y)$, which implies that $e^{x+y} = e^x e^y$.

Differentiation: $\frac{d}{dx} e^x = e^x$

proof: Let $f(x) = \ln(x)$, $f^{-1}(x) = e^x$. Then by Section 6.1,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(e^x)} = \frac{1}{\frac{1}{e^x}} = e^x$$

Examples: Differentiate

$$\textcircled{1} \quad y = e^{5x+2}; \quad y' = e^{5x+2} \cdot (5x+2)' = 5e^{5x+2}$$

$$\textcircled{2} \quad y = xe^{x^2}; \quad y' = (x)'e^{x^2} + x(e^{x^2})' = e^{x^2} + x \cdot e^{x^2} \cdot 2x$$

In general, $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$

Integration: $\int e^x dx = e^x + C$

Examples: ① $\int (x^2+1) e^{\frac{x^3+3x}{3}} dx$. Let $u = \frac{x^3+3x}{3}$, $du = (x^2+1) dx$
 $\int e^u du = e^u + C = e^{\frac{x^3+3x}{3}} + C$

② $\int 2e^{\cos x} \cdot \sin(x) dx$. Let $u = \cos x$, $-du = \sin x dx$

$$-2 \int e^u du = -2e^u + C = -2e^{\cos x} + C$$